THE FIRST HOCHSCHILD COHOMOLOGY GROUP OF A CLUSTER-TILTED ALGEBRA REVISITED

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Dedicated to Christophe Reutenauer for his 60$^{\text{th}}$ birthday.

ABSTRACT. Given a cluster-tilted algebra B we study its first Hochschild cohomology group $\mathsf{HH}^1(B)$ with coefficients in the B-B-bimodule B. If C is a tilted algebra such that B is the relation extension of C by $E = \mathsf{Ext}^2_C(DC, C)$, then we prove that $\mathsf{HH}^1(B)$ is isomorphic, as a vector space, to the direct sum of $\mathsf{HH}^1(C)$ with $\mathsf{HH}^1(B, E)$. This yields homological interpretations for results of the first and the fourth author with M.J. Redondo.

INTRODUCTION

This paper is the third of a series devoted to studying the first Hochschild cohomology group of a cluster-tilted algebra [6, 5].

Cluster-tilted algebras appeared naturally during the study of the cluster algebras of Fomin and Zelevinsky [16]. They were introduced in [10] and independently in [11] for the type $\mathbb{A}$ and, since then, have been the subject of several investigations. In particular, it was proved in [1] that if C is a tilted algebra, then the trivial extension of C by the C-C-bimodule $E = \mathsf{Ext}^2_C(DC, C)$, called the *relation-extension* of C, is cluster-tilted and, conversely, every cluster-tilted algebra arises in this way.

The Hochschild cohomology groups of an algebra were defined by Hochschild in 1945, see [19]. These are subtle homological invariants, not only of the algebra, but also of its derived category [17, 20]. In [24], the vanishing of the first Hochschild cohomology group was related to the simple connectedness of the algebra. Further connections between the first Hochschild cohomology group and the fundamental groups of an algebra were obtained in [4, 14]. It was then a natural question to try to relate the first Hochschild cohomology group $\mathsf{HH}^1(C)$ of a tilted algebra C with coefficients in the C-C-bimodule C to the corresponding group $\mathsf{HH}^1(B)$ of the relation-extension B. For this purpose, a first observation is that, because B is a trivial extension of C, then there exists a canonical morphism $\varphi \colon \mathsf{HH}^1(B) \to \mathsf{HH}^1(C)$, see [6]. Next, an equivalence relation was defined in [6] between the arrows in the quiver of B which are not in the quiver of C. The number of equivalence classes is then denoted by $n_{B,C}$. It was shown in [6] that if C is a tilted algebra over an algebraically closed field $\mathbb{k}$ such that the relation-extension B is schurian, then there exists a short exact sequence of vector spaces

$$0 \longrightarrow \mathbb{k}^{n_{B,C}} \longrightarrow \mathsf{HH}^1(B) \overset{\varphi}{\longrightarrow} \mathsf{HH}^1(C) \longrightarrow 0.$$

1

This result was generalised in [5] to the cases where C is constricted (in the sense of [9]) or B is tame. The proofs of these two results were combinatorial. In the case of representation finite cluster tilted-algebras, the Hochschild cohomology has also been computed by Ladkani using different methods see [22].

Our objective in this note is to provide a homological interpretation of this short exact sequence, removing all assumptions on B or C. Our main theorem may be stated as follows.

Theorem. *Let $\Bbbk$ be an algebraically closed field , C be a tilted $\Bbbk$-algebra and B be the trivial extension of C by $E = \mathsf{Ext}_C^2(DC, C)$. Then there exists a short exact sequence of vector spaces*

$$0 \longrightarrow \mathsf{HH}^1(B, E) \longrightarrow \mathsf{HH}^1(B) \xrightarrow{\varphi} \mathsf{HH}^1(C) \longrightarrow 0.$$

The proof of this theorem is largely homological and different from those in [6, 5]. We also prove that $\dim_\Bbbk \mathsf{HH}^1(B, E) \geqslant n_{B,C}$ and equality holds if and only if the indecomposable summands of the C-C-bimodule E are orthogonal bricks. Clearly, this is the case under the hypotheses of [6, 5].

Our paper is organised as follows. After a short preliminary section 1, we start the proof of our theorem in section 2 by proving the left exactness of the required sequence. It is next shown in section 3 to be right exact and we study the kernel of the map φ in section 4. We end the paper with an example in section 5.

1. Preliminaries

1.1. **Quivers and relations.** While we briefly recall some concepts concerning bound quivers and algebras, we refer the reader to [7] or [8], for instance, for unexplained notions.

Let $\Bbbk$ be a commutative field. A *quiver* Q is the data of two sets, Q_0 (the *vertices*) and Q_1 (the *arrows*) and two maps $s, t \colon Q_1 \to Q_0$ that assign to each arrow α its *source* $s(\alpha)$ and its *target* $t(\alpha)$. We write $\alpha \colon s(\alpha) \to t(\alpha)$. If $\beta \in Q_1$ is such that $t(\alpha) = s(\beta)$ then the composition of α and β is the *path* $\alpha\beta$. This extends naturally to paths of arbitrary positive length. The *path algebra* $\Bbbk Q$ is the $\Bbbk$-algebra whose basis is the set of all paths in Q, including one stationary path e_x at each vertex $x \in Q_0$, endowed with the multiplication induced from the composition of paths. If $|Q_0|$ is finite, the sum of the stationary paths is the identity.

In case $\Bbbk$ is algebraically closed, then any finite-dimensional basic and connected algebra A can be obtained as a quotient of a path algebra $A \simeq \Bbbk Q/I$. In this case, the pair (Q, I) is called a *bound quiver*. Given two vertices $x, y \in Q_0$, a *relation* from x to y is a $\Bbbk$ - linear combination $r = \sum_{i=1}^m \lambda_i w_i \in e_x I e_y$ of paths w_i of length at least two from x to y. The relation r is *minimal* if none of the scalars λ_i is zero, and for any proper subset $J \subset \{1, \dots, m\}$ one has $\sum_{j \in J} \lambda_j w_j \notin e_x I e_y$. The relation r is said to be *strongly minimal* if, as before, $\lambda_j \neq 0$ and for any proper subset $J \subset \{1, \dots, m\}$ there is no family of non-zero scalars μ_j such that $\sum_{j \in J} \mu_j w_j \in e_x I e_y$.

Given an algebra $A \simeq \Bbbk Q/I$, a *system of relations* for an algebra A is a subset $\mathcal{R}$ of $\bigcup_{x,y \in Q_0} e_x I e_y$ that generates I as a two-sided ideal, but such that no proper subset of $\mathcal{R}$ does. It is shown in [5, 2.2] that one may assume $\mathcal{R}$ to be a system of strongly minimal relations.

1.2. **Cluster-tilted algebras.** Let H be a finite dimensional hereditary $\Bbbk$-algebra, $\mathsf{mod}\text{-}H$ the category of finite dimensional right H-modules and $\mathcal{D}^b(\mathsf{mod}\text{-}H)$ the corresponding bounded derived category. It is a triangulated category with shift functor denoted by [1], and it has an Auslander-Reiten translation τ. The *cluster category* of H is the orbit category $\mathcal{C}_H := \mathcal{D}^b(\mathsf{mod}\text{-}H)/\tau^{-1} \circ [1]$. Again, it is a triangulated category having almost split triangles. An object T in $\mathcal{C}_H$ is a (basic) *tilting* object if $\mathsf{Ext}^1_{\mathcal{C}_H}(T, T) = 0$ and T is the sum of $\mathsf{rk}\, \mathsf{K}_0(H)$ indecomposable objects which

in addition are pairwise non-isomorphic. The endomorphism algebra $\mathsf{End}_{\mathcal{C}_H}(T)$ is a *cluster-tilted algebra*.

Consider a tilting module U over a hereditary algebra H, so that the algebra $C = \mathsf{End}_H(U)$ is a *tilted algebra* [18] and denote by D the standard duality $\mathsf{Hom}_{\Bbbk}(-, \Bbbk)$ between $\mathsf{mod}\text{-}H$ and $\mathsf{mod}\text{-}H^{op}$. Let E be the C-C-bimodule $E = \mathsf{Ext}^2_C(DC, C)$ with the natural actions. The trivial extension $C \ltimes E$ of C by E, called the *relation-extension* of C, is the algebra whose underlying vector space is $C \oplus E$, endowed with the multiplication induced by the bimodule structure of E, namely

$$(c_1, e_1) \cdot (c_2, e_2) = (c_1 c_2, c_1 e_2 + e_1 c_2).$$

It was shown in [1] that $B = C \ltimes E$ is a cluster-tilted algebra, and, conversely, every cluster-tilted algebra arises in this way, though not uniquely.

Also, the natural projection $p \colon B \to C$ is a morphism of algebras, and so is its right inverse $q \colon C \to B$. We have a short exact sequence of B-B-bimodules

$$(1) \qquad\qquad 0 \longrightarrow E \xrightarrow{\ i\ } B \xrightarrow{\ p\ } C \longrightarrow 0.$$

Also, it was shown in [1] that once the bound quiver (Q, I) of C is known then that of B, say $(\tilde{Q}, \tilde{I})$, is obtained as follows:

- $\tilde{Q}_0 = Q_0$;
- For $x, y \in Q_0$, the set of arrows in $\tilde{Q}$ from x to y equals the set of arrows in Q from x to y (which we call *old arrows*) plus $|\mathcal{R} \cap e_y I e_x|$ additional arrows (which we call *new arrows*).

The relations defining $\tilde{I}$ are given by the partial cyclic derivatives of the potential $W = \sum_{r \in \mathcal{R}} \gamma_r r$, where γ_r is the new arrow associated to the relation r (see [21]). Potentials are considered up to cyclic permutations : two potentials are *cyclically equivalent* if their difference lies in the linear span of all elements of the form $\alpha_1 \alpha_2 \cdots \alpha_j - \alpha_j \alpha_1 \cdots \alpha_{j-1}$, where $\alpha_1 \alpha_2 \cdots \alpha_j$ is an oriented cycle. We recall from [15] that, for a given arrow β the *cyclic partial derivative* ∂_β of W is defined on each cyclic summand $\beta_1 \cdots \beta_s$ by $\partial_\beta(\beta_1 \cdots \beta_s) = \sum_{i:\beta=\beta_i} \beta_{i+1} \cdots \beta_s \beta_1 \cdots \beta_{i-1}$. In particular, the cyclic derivative is invariant under cyclic permutation.

1.3. Hochschild cohomology.

We recall some notions concerning Hochschild cohomology, but for unexplained ones, we refer to [17, 23] for instance. Given a $\Bbbk$-algebra A, let $A^e = A \underset{\Bbbk}{\otimes} A^{op}$ be its enveloping algebra. It is well-known that the category of A-A-bimodules is equivalent to that of A^e-modules. If $_A X_A$ is a bimodule, then the Hochschild cohomology groups of A with coefficients in X are the extension groups $\mathsf{HH}^i(A, X) = \mathsf{Ext}^i_{A^e}(A, X)$. In case $X = A$, we simply write $\mathsf{HH}^i(A)$.

We are interested in computing the Hochschild cohomology groups of cluster-tilted algebras, which are given by quivers and relations. In this context we can use a convenient resolution for computing the Ext groups, see [12, 1.1 and 1.2]. Let $A = \Bbbk Q/I$ and $\mathfrak{r}$ be its Jacobson radical. Then $A_0 = A/\mathfrak{r}$ is the semisimple algebra generated by the vertices of Q, and as A_0-bimodules one has $A = A_0 \oplus \mathfrak{r}$. The following result will be used in the sequel.

1.4. Proposition (1.2 in [12]).

Given an A^e-module X, the Hochschild cohomology groups $\mathsf{HH}^i(A, X)$ are the cohomology groups of the complex

$$0 \longrightarrow X^{A_0} \xrightarrow{\ d^1\ } \mathsf{Hom}_{A_0^e}(\mathfrak{r}, X) \xrightarrow{\ d^2\ } \mathsf{Hom}_{A_0^e}(\mathfrak{r} \otimes \mathfrak{r}, X) \xrightarrow{\ d^3\ } \mathsf{Hom}_{A_0^e}(\mathfrak{r}^{\otimes^3}, X) \longrightarrow \cdots$$

where the tensor products are taken over A_0, $X^{A_0} = \{x \in X | sx = xs, \text{ for all } s \in A_0\} =$
$\bigoplus_{s \in Q_0} e_s X e_s$, *the differentials are given by* $(d^1 x)(r) = xr - rx$, *and, in general for* $i \geqslant 2$

$$
\begin{aligned}
d^i f(r_1 \otimes \cdots \otimes r_i) &= r_1 f(r_2 \otimes \cdots \otimes r_i) + \sum_{j=1}^{i-1} (-1)^j f(r_1 \otimes \cdots \otimes r_j r_{j+1} \otimes \cdots r_i) \\
&+ (-1)^i f(r_1 \otimes \cdots \otimes r_{i-1}) r_i.
\end{aligned}
$$

$\square$

From this, one sees that $\mathsf{HH}^0(A, X) = \{x \in X | ax = xa, \text{ for all } a \in A\}$, so in particular $\mathsf{HH}^0(A)$ is the centre of A. The kernel of the map d^2 is the set of A_0-bilinear maps $f \colon \mathfrak{r} \to X$ such that $f(a_1 a_2) = a_1 f(a_2) + f(a_1) a_2$ for $a_1, a_2 \in \mathfrak{r}$, that is, the *derivations* of $\mathfrak{r}$ in X. If we extend such a derivation f to A_0 by letting $f(A_0) = 0$, we obtain the derivations of A in X (see [13, 23]). Also, for a fixed $x \in X$ the map $d^1 x = [x, -] \colon a \mapsto ax - xa$ is a derivation, and $\mathsf{Im}\, d^1$ is the set of *inner* derivations.

A useful feature of the complex above is that we only need to deal with maps that are A_0-bilinear. Thus, if $r \in e_i \mathfrak{r} e_j$, then $f(r) = f(e_i r e_j) = e_i f(r) e_j \in e_i X e_j$.

1.5. **Remark.** Alternatively, derivations can be described as follows. Let $\Bbbk Q_1$ be the A_0-bimodule generated by the set of arrows of Q. A $\Bbbk Q_0$-bilinear map $\delta \colon \Bbbk Q_1 \to \Bbbk Q$ can be extended to a $\Bbbk Q_0$-bilinear map $\Bbbk Q \to \Bbbk Q$ using the Leibniz rule, so it becomes a derivation of $\Bbbk Q$. Then, the map induces a unique derivation of A if and only if $\delta(I) \subseteq I$.

2. A LEFT EXACT SEQUENCE OF COHOMOLOGY GROUPS

Let C be a tilted algebra, $E = \mathsf{Ext}^2_C(DC, C)$ and $B = C \ltimes E$ the corresponding cluster-tilted algebra. Upon applying the functor $\mathsf{Hom}_{B^e}(B, -)$ to the short exact sequence (1) of section 1.2 we obtain a long exact sequence of cohomology groups

$$0 \to \mathsf{Hom}_{B^e}(B, E) \to \mathsf{Hom}_{B^e}(B, B) \to \mathsf{Hom}_{B^e}(B, C) \to \mathsf{Ext}^1_{B^e}(B, E) \xrightarrow{\iota} \mathsf{Ext}^1_{B^e}(B, B) \xrightarrow{\overline{p}} \mathsf{Ext}^1_{B^e}(B, C) \to \cdots$$

Our first task is to compare the cohomology groups of C, that is $\mathsf{HH}^i(C)$, to those of B with coefficients in C, that is $\mathsf{HH}^i(B, C) = \mathsf{Ext}^i_{B^e}(B, C)$. The following lemma will be useful in the sequel.

2.1. **Lemma.**
a) *There is an isomorphism* $\mathsf{HH}^0(C) \simeq \mathsf{HH}^0(B, C)$,
b) *There is a monomorphism* $\mathsf{HH}^1(C) \hookrightarrow \mathsf{HH}^1(B, C)$.

Proof. We use proposition 1.4. Let $\mathfrak{r}_C$ and $\mathfrak{r}_B$ be the radicals of C and B, respectively. Because $B = C \ltimes E$ is a trivial extension, we have $\mathfrak{r}_B = \mathfrak{r}_C \oplus E$. In particular, the projection $p \colon B \to C$ restricts to a natural retraction $\mathfrak{r}_B \to \mathfrak{r}_C$ which we still denote by p. Further, we let $q \colon \mathfrak{r}_C \to \mathfrak{r}_B$ be its right inverse (section), and $-^* = \mathsf{Hom}_{C_0^e}(-, C)$.

Note that we have $B_0 = C_0$. The groups $\mathsf{Ext}^i_{C^e}(C, C)$ are the cohomology groups of the upper complex in the diagram below, whereas $\mathsf{Ext}^i_{B^e}(B, C)$ are those of the lower one.

$$
\begin{array}{ccccccc}
0 \longrightarrow & C^{C_0} & \xrightarrow{d^1_C} & \mathsf{Hom}_{C^e_0}(\mathfrak{r}_C, C) & \xrightarrow{d^2_C} & \mathsf{Hom}_{C^e_0}(\mathfrak{r}_C \otimes \mathfrak{r}_C, C) & \longrightarrow \cdots \\[2mm]
& \| & & \downarrow{p^*} & & \downarrow{(p \otimes p)^*} & \\[2mm]
0 \longrightarrow & C^{C_0} & \xrightarrow{d^1_B} & \mathsf{Hom}_{C^e_0}(\mathfrak{r}_B, C) & \xrightarrow{d^2_B} & \mathsf{Hom}_{C^e_0}(\mathfrak{r}_B \otimes \mathfrak{r}_B, C) & \longrightarrow \cdots
\end{array}
$$

The map p is a retraction, hence p^* is injective and we have

$$\mathsf{HH}^0(C) = \mathsf{Ext}^0_{C^e}(C, C) = \mathsf{Ker}\, d^1_C = \mathsf{Ker}\, p^* d^1_C = \mathsf{Ker}\, d^1_B = \mathsf{Ext}^0_{B^e}(B, C) = \mathsf{HH}^0(B, C).$$

which proves a). Moreover, we have $\mathsf{Im}\, d^1_B = \mathsf{Im}\, (p^* d^1_C) \simeq \mathsf{Im}\, d^1_C$. Because $(p \otimes p)^*$ is also injective, we also have

$$\mathsf{Ker}\, d^2_C = \mathsf{Ker}\, [(p \otimes p)^* d^2_C] = \mathsf{Ker}\, (d^2_B p^*)$$

Now p^*, being injective, induces an obvious monomorphism $\mathsf{Ker}\, d^2_B p^* \to \mathsf{Ker}\, d^2_B$ given by $x \mapsto p^*(x)$. Hence we have a monomorphism

$$\mathsf{HH}^1(C) = \frac{\mathsf{Ker}\, d^2_C}{\mathsf{Im}\, d^1_C} \longhookrightarrow \frac{\mathsf{Ker}\, d^2_B}{\mathsf{Im}\, d^1_B} = \mathsf{HH}^1(B, C)$$

and this proves b).

$\square$

Now, we recall from [6, 5] that there exists a natural map $\varphi \colon \mathsf{HH}^1(B) \to \mathsf{HH}^1(C)$ defined as follows: any $\Bbbk$-linear map $\delta \colon B \to B$ induces a $\Bbbk$-linear map $p\delta q \colon C \to C$. A direct computation shows that if δ is a derivation (or an inner derivation), then so is $p\delta q$. The map φ sends the class of δ to that of $p\delta q$. In homological terms, φ is expressed as

$$\varphi = \mathsf{Ext}^1_{B^e}(q, C)\mathsf{Ext}^1_{B^e}(B, p).$$

In order to simplify notation, we write $\overline{p} = \mathsf{Ext}^1_{B^e}(B, p)$ and $\tilde{q} = \mathsf{Ext}^1_{B^e}(q, C)$. We need the following lemma.

2.2. **Lemma.** $\mathsf{Ker}\, \varphi = \mathsf{Ker}\, \overline{p}$.

Proof. Clearly $\varphi = \tilde{q}\,\overline{p}$ implies immediately that $\mathsf{Ker}\, \overline{p} \subseteq \mathsf{Ker}\, \varphi$. Let thus δ be a derivation whose class belongs to $\mathsf{Ker}\, \varphi$. Thus $p\delta q$ is an inner derivation of C, that is there exists $c \in C$ such that $p\delta q = [c, -]$. Write $\delta_c = [c, -]$. Replacing δ by $\delta - \delta_c$ we can assume that $p\delta q = 0$, that is $f = p\delta$ equals zero when restricted to C. Now f being a derivation on B which is zero on C, is a morphism of C-C-bimodules $f \colon E \to C$. Indeed, let $e \in E$ and $c \in C$ then $f(ec) = ef(c) + f(e)c = f(e)c$, and similarly $f(ce) = cf(e)$. Let now $\gamma \colon i \to j$ be a new arrow (thus, a generator of E as C-C-bimodule) then f sends $\gamma \in e_i E e_j$ into $e_i C e_j$ which is zero, because C is triangular. Therefore, $f = 0$ on E, so $\mathsf{Ker}\, \varphi = \mathsf{Ker}\, \overline{p}$. $\square$

The next step is to extract a left exact sequence from the long exact cohomology sequence involving only the degree one terms. In order to avoid cumbersome notations, let $\iota = \mathsf{Ext}^1_{B^e}(B, i)$ where $i \colon E \hookrightarrow B$ is the inclusion.

2.3. Corollary. *There is an exact sequence*

$$0 \longrightarrow \mathsf{HH}^1(B,E) \xrightarrow{\ \iota\ } \mathsf{HH}^1(B) \xrightarrow{\ \varphi\ } \mathsf{HH}^1(C)$$

with $\varphi = \tilde{q}\,\overline{p}$.

Proof. Because C is connected and triangular, its centre is isomorphic to $\Bbbk$. Lemma 2.1 then gives $\mathsf{Hom}_{B^e}(B,C) \simeq \Bbbk$. In addition, the map $\mathsf{Hom}_{B^e}(B,p) : \mathsf{Hom}_{B^e}(B,B) \to \mathsf{Hom}_{B^e}(B,C)$ appearing in the long exact sequence of cohomology groups is not zero, because the identity (on C) belongs to its image. Thus, this map is surjective, and hence ι is injective. Using the notation of Lemma 2.2, we have an exact sequence

$$0 \longrightarrow \mathsf{HH}^1(B,E) \xrightarrow{\ \iota\ } \mathsf{HH}^1(B) \xrightarrow{\ \overline{p}\ } \mathsf{HH}^1(B,C)$$

and a map $\tilde{q}\colon \mathsf{HH}^1(B,C) \to \mathsf{HH}^1(C)$ such that $\varphi = \tilde{q}\,\overline{p}$. Invoking Lemma 2.2 completes the proof. $\qquad\square$

3. The surjectivity of φ

Our next step is to show that φ is surjective (as was shown in [6, 5] under some additional hypotheses). Thus given a derivation δ of C, we want to extend it to a derivation $\tilde{\delta}$ of B. We proceed in two steps:

(1) First of all, we consider δ as a $\Bbbk Q_0$-bilinear derivation from $\Bbbk Q$ to itself that sends each relation $r \in \mathcal{R}$ to I. We extend δ to a map $\tilde{\delta}\colon \Bbbk\tilde{Q} \to \Bbbk\tilde{Q}$ by defining it on the new arrows. Extending it by using the Leibniz rule we obtain a derivation of $\Bbbk\tilde{Q}$. We then show that $\tilde{\delta}$ vanishes (up to cyclic permutation) on the potential W;

(2) Finally we show that in fact $\tilde{\delta}(\tilde{I}) \subseteq \tilde{I}$ so that in fact $\tilde{\delta}$ is a derivation of B.

We recall that two paths u, v in a quiver are said to be *parallel* if $s(u) = s(v)$ and $t(u) = t(v)$, and *antiparallel* if $s(u) = t(v)$ and $t(u) = s(v)$.

We need to define δ on the new arrows, which, as already observed, are in bijection with the elements of $\mathcal{R}$. Let $r_i \in \mathcal{R}$ be a relation from x to y in $\Bbbk Q$. Since $\delta(r_i) \in e_x I e_y$, there exist scalars b_{ik}, paths u_{ik}, v_{ik} and relations r_{j_k} such that

$$\delta(r_i) = \sum_{k=1}^{m} b_{ik} u_{ik} r_{j_k} v_{ik}. \tag{2}$$

Since r_i is parallel to $u_{ik} r_{j_k} v_{ik}$, and r_i is anti-parallel to the corresponding new arrow γ_i, then each $v_{ik} \gamma_i u_{ik}$ is a path anti-parallel to r_{j_k}.
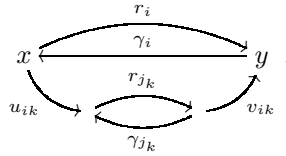
For a given arrow γ_j we want to collect all the terms $v_{ik} r_i u_{ik}$ where $\gamma_j = \gamma_{j_k}$ with r_{j_k} appearing in the expression of $\delta(r_i)$ (equation 2, above). More precisely, for a fixed new arrow γ_j define

$$\mathcal{E}_j = \{(i,k)\mid r_{j_k} = r_j \text{ in the expression of } \delta(r_i)\}$$

3.1. **Lemma.** *Let $\delta\colon \Bbbk Q \to \Bbbk Q$ be a derivation such that $\delta(I) \subseteq I$. Then the map $\tilde{\delta}\colon \Bbbk\tilde{Q} \to \Bbbk\tilde{Q}$ defined on the arrows of $\tilde{Q}$ by*

$$
\tilde{\delta}(\alpha) = \begin{cases} \delta(\alpha) & \text{if } \alpha \text{ is an old arrow;} \\ -\displaystyle\sum_{(i,k)\in\mathcal{E}_j} b_{ik}v_{ik}\gamma_i u_{ik} & \text{if } \alpha = \gamma_j. \end{cases}
$$

and extended by the Leibniz rule is a derivation of $\Bbbk\tilde{Q}$, satisfying $\tilde{\delta}(W) = 0$ up to cyclic permutation.

Proof. The only thing we have to prove is that $\tilde{\delta}(W)$ is zero up to cyclic permutation, but this follows readily from:

$$
\begin{aligned}
\tilde{\delta}(W) &= \sum_{i=1}^{n} \tilde{\delta}(r_i\gamma_i) \\
&= \sum_{i=1}^{n} \delta(r_i)\gamma_i + \sum_{j=1}^{n} r_j\tilde{\delta}(\gamma_j) \\
&= \sum_{i=1}^{n}\sum_{k=1}^{m} b_{ik}u_{ik}r_k v_{ik}\gamma_i + \sum_{j=1}^{n} r_j \sum_{(i,k)\in\mathcal{E}_j} -b_{ik}v_{ik}\gamma_i u_{ik} \\
&= \sum_{i=1}^{n}\sum_{k=1}^{m} (b_{ik}u_{ik}r_k v_{ik}\gamma_i - b_{ik}r_i v_{ik}\gamma_i u_{ik})
\end{aligned}
$$

which is zero up to cyclic permutation. $\square$

Let us now show that $\tilde{\delta}(\tilde{I}) \subseteq \tilde{I}$. The ideal $\tilde{I}$ is generated by the partial derivatives $\partial_\alpha W$ of the potential W with respect to the arrows of $\tilde{Q}$. If γ is a new arrow, then $\partial_\gamma(W)$ is an old relation (thus an element of I), and then $\tilde{\delta}(\partial_\gamma W) = \delta(\partial_\gamma W) \in I \subseteq \tilde{I}$, since δ is a derivation of C. Thus we only need to look at the partial derivatives with respect to the old arrows, or, equivalently, we only need to look at the new relations. Before proving the desired result we need some preliminary observations.

Assume α is an old arrow, starting at a vertex i. Then define:

$$
\begin{aligned}
W_1 &= \text{ sum of all terms in } W \text{ that pass through } i \\
&= \sum_{\gamma : s(\gamma)=i} \gamma \, \partial_\gamma W;
\end{aligned}
$$

$$
\begin{aligned}
W_2 &= \text{ sum of all terms in } W \text{ that do not pass through } i \text{ and contain an arrow} \\
&\quad\ \beta \text{ such that } \tilde{\delta}(\beta) \text{ passes through } i;
\end{aligned}
$$

$$
W_3 = \text{ sum of all other terms in } W.
$$

Since the quiver Q has no oriented cycles, we can number its vertices in such a way that for every arrow $\alpha \in Q_1$ we have $s(\alpha) < t(\alpha)$. We fix such a numbering in the sequel.

3.2. **Lemma.** *Let Θ be the set of all the arrows β appearing in W_2 such that $\tilde{\delta}(\beta)$ passes through i. Then $W_2 = \displaystyle\sum_{\beta\in\Theta} \beta\partial_\beta W$.*

Proof. We show that a cycle w appearing in W_2 contains exactly one arrow β such that $\tilde{\delta}(\beta)$ passes through the vertex i. Assume the contrary, that is, there are two such arrows, say β and β'. By [6, 2.1] one of them must be an old arrow, say β. It then follows from the definition of $\tilde{\delta}$ that $\tilde{\delta}(\beta) = \delta(\beta)$ is a linear combination of paths containing only old arrows, which go from $s(\beta)$ to $t(\beta)$, and at least one of them passes through i, so that $s(\beta) < i < t(\beta)$. We now show that the cycle w can contain at most one old arrow having this property. Assume to the contrary that, up to cyclic permutation, $w = \beta u \beta' u'$ with u, u' paths in Q, and β, β' two arrows such that $\tilde{\delta}(\beta) = b_1 b_2 +$ other terms, $\tilde{\delta}(\beta') = b_1' b_2' +$ other terms, with b_1, b_1' paths ending at i.
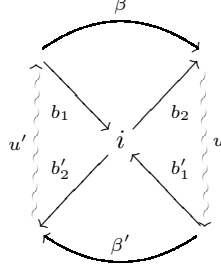
Suppose first that β' is an old arrow. Then b_1' and b_2' are paths in Q. Since $w = \beta u \beta' u'$ is a summand of the potential, it contains exactly one new arrow. If this new arrow occurs in u', then $b_2 u b_1'$ is a cycle in $\widetilde{Q}$ consisting of old arrows, a contradiction. If the new arrow appears in u, a similar argument works. Thus β' is a new arrow and we have by definition,

$$\tilde{\delta}(\beta') = -\sum_j b_j v_j \gamma_j u_j,$$

where γ_j are new arrows, v_j, u_j are old paths and $v_j \gamma_j u_j$ is a path from $s(\beta')$ to $t(\beta')$. But

$$t(\beta') \leqslant s(\beta) < i < t(\beta) \leqslant s(\beta')$$

so $v_j \gamma_j u_j$ cannot pass through i, a contradiction.

We can thus write $W_2 = \sum_{\beta \in \Theta} \beta w_\beta$, where each w_β is a linear combination of paths. By definition of W_2, w_β cannot pass through i, hence all the terms of W which contain an arrow $\beta \in \Theta$ appear exactly once in this sum. Therefore $w_\beta = \partial_\beta W$. $\qquad\square$

We are now able to show that for every old arrow α we have $\tilde{\delta}\,(\partial_\alpha W) \in \widetilde{I}$. We do so in two steps.

For a fixed vertex $i \in Q_0$ let $\widetilde{I}_{\neq i}$ be the ideal of $\Bbbk\widetilde{Q}$ generated by all the partial derivatives $\partial_\varepsilon W$ such that ε does not end at i.

3.3. Lemma. *If $\sum_{\gamma:s(\gamma)=i} \gamma w_\gamma$ is a linear combination of cycles which belongs to $\widetilde{I}_{\neq i}$, then $w_\gamma \in \widetilde{I}_{\neq i}$ for every γ such that $s(\gamma) = i$.*

Proof. Write $\sum_{\gamma:s(\gamma)=i} \gamma w_\gamma = \sum_j b_j u_j r_j v_j$ with b_j scalars, r_j generators of $\widetilde{I}_{\neq i}$ and u_j, v_j paths. Note that by definition the first arrow of the u_j is one of the arrows γ starting at i, thus $u_j = \gamma_j u_j'$. Then

$$\sum_{\gamma:s(\gamma)=i} \gamma w_\gamma = \sum_j b_j \gamma_j u_j' r_j v_j$$

and, upon comparing the terms, we obtain

$$w_\gamma = \sum_{j:\gamma=\gamma_j} b_j u'_j r_j v_j \in \widetilde{I}_{\neq i}.$$

$\square$

We can now prove the required statement.

3.4. Lemma. *Let α be an old arrow, and $\tilde{\delta}$ defined as before. Then $\tilde{\delta}(\partial_\alpha W) \in \widetilde{I}$.*

Proof. We showed in 3.1 that up to cyclic permutation $\tilde{\delta}(W) = 0$. Take any $\alpha \in Q_1$ and let $i = s(\alpha)$. Then by construction we have $W = W_1 + W_2 + W_3$. Thus using the Leibniz rule, we obtain (up to cyclic permutations):

$$(3) \qquad 0 = \sum_{\gamma:s(\gamma)=i} \tilde{\delta}(\gamma)\partial_\gamma W + \sum_{\gamma:s(\gamma)=i} \gamma\tilde{\delta}(\partial_\gamma W) + \sum_{\beta\in\Theta} \tilde{\delta}(\beta)\partial_\beta W + \sum_{\beta\in\Theta} \beta\tilde{\delta}(\partial_\beta W) + \tilde{\delta}(W_3).$$

By definition, the terms $\partial_\gamma W$ and $\partial_\beta W$ appearing in the expression above belong to $\widetilde{I}_{\neq i}$. Moreover, the terms in the first two sums involve paths passing through i, and the last term, as well as the fourth sum consist of paths not passing through i. Concerning the third sum, each $\tilde{\delta}(\beta)$ is a linear combination of paths, some of them (at least one) passing through i, and some (may be none) not passing through i. Accordingly, for each β collect the terms passing through i together, and call the result $\tilde{\delta}_i(\beta)$. Collect the remaining terms to obtain $\tilde{\delta}_0(\beta)$. Thus the third sum above is

$$\sum_{\beta\in\Theta} \left(\tilde{\delta}_i(\beta) + \tilde{\delta}_0(\beta) \right) \partial_\beta W.$$

Altogether $\tilde{\delta}(W)$ splits into the sum of terms passing through i and the sum of terms not passing through i; and both sums are equal to zero. Therefore we have the following equation of cyclic words.

$$(4) \qquad 0 = \sum_{\gamma:s(\gamma)=i} \tilde{\delta}(\gamma)\,\partial_\gamma W + \sum_{\gamma:s(\gamma)=i} \gamma\,\tilde{\delta}(\partial_\gamma W) + \sum_{\beta\in\Theta} \tilde{\delta}_i(\beta)\,\partial_\beta W.$$

Furthermore, we can view these cyclic words as paths starting at the vertex i. Then, the sum of these paths equals 0. Since each term $\partial_\gamma W$ and $\partial_\beta W$ belongs to $\widetilde{I}_{\neq i}$, so does $\displaystyle\sum_{\gamma:s(\gamma)=i} \gamma\tilde{\delta}(\partial_\gamma W)$. By Lemma 3.3 this implies that $\tilde{\delta}(\partial_\gamma W) \in \widetilde{I}$ for each γ with $s(\gamma) = i$. So, $\tilde{\delta}(\partial_\alpha W) \in \widetilde{I}$. $\square$

From the preceding lemmata follows the theorem.

3.5. Theorem. *With the notations of section 2 there is an exact sequence*

$$0 \longrightarrow \mathsf{HH}^1(B,E) \overset{\iota}{\longrightarrow} \mathsf{HH}^1(B) \overset{\varphi}{\longrightarrow} \mathsf{HH}^1(C) \longrightarrow 0 \,.$$

$\square$

We have an immediate consequence of this theorem. We recall first that if B is the relation extension of C, then C is not uniquely determined by B (see, for instance, [2]).

3.6. Corollary. *Let C, C' be tilted algebras and $E = \mathsf{Ext}^2_C(DC, C), E' = \mathsf{Ext}^2_{C'}(DC', C')$. If $B = \mathcal{C} \ltimes E \simeq \mathcal{C}' \ltimes E'$, then $\mathsf{HH}^1(C) \simeq \mathsf{HH}^1(C')$ and $\mathsf{HH}^1(B,E) \simeq \mathsf{HH}^1(B,E')$.*

Proof. Under the stated hypothesis, C and C' are tilted algebras of the same type (see [2]). Because of [17, 4.2] we have $\mathsf{HH}^1(C) \simeq \mathsf{HH}^1(C')$. The theorem then implies immediately that we also have $\mathsf{HH}^1(B, E) \simeq \mathsf{HH}^1(B, E')$. □

4. Interpretation of the Kernel

In this section we proceed to relate our result to those of [6, 5]. There, under some hypotheses on B or C, the kernel of the map $\varphi \colon \mathsf{HH}^1(B) \to \mathsf{HH}^1(C)$ was computed by means of an equivalence relation on the set of new arrows which we now describe.

Given a strongly minimal relation $\sum_{i=1}^m a_i w_i$ in $\widetilde{I}$, either it is a relation of I, or there exist exactly m new arrows $\gamma_1, \ldots, \gamma_m$ and old paths u_i, v_i such that $w_i = u_i \gamma_i v_i$ (see [6, 3.1] or [5, 3.1]). We let $\approx$ be the smallest equivalence relation on the set of new arrows such that $\gamma \approx \gamma'$ whenever γ and γ' are two new arrows appearing in a strongly minimal relation. Finally, we let $n = n_{B,C}$ be the number of equivalence classes for $\approx$.

4.1. Corollary. *Let $B = C \ltimes E$ be such that B is tame or C is constricted. Then we have* $\mathsf{HH}^1(B, E) \simeq \Bbbk^n$.

Proof. This follows immediately from Theorem 3.5 and the main result of [5]. □

We now prove that $\mathsf{End}_{C^e}(E)$ is always a subspace of $\mathsf{HH}^1(B, E)$. We recall that $\mathsf{HH}^1(B, E)$ is $\mathsf{Ker}\, d^2 / \mathsf{Im}\, d^1$ in the complex below

$$0 \longrightarrow E^{B_0} \xrightarrow{\ d^1\ } \mathsf{Hom}_{B_0^e}(\mathfrak{r}_B, E) \xrightarrow{\ d^2\ } \mathsf{Hom}_{B_0^e}(\mathfrak{r}_B \otimes \mathfrak{r}_B, E) \longrightarrow \cdots$$

4.2. Lemma. *We have inclusions of vector spaces*

$$\mathsf{End}_{C^e}(E) \subseteq \mathsf{HH}^1(B, E) \subseteq \mathsf{End}_{C^e}(E) \oplus \frac{\mathsf{Hom}_{C_0^e}(\mathfrak{r}_C, E)}{\mathsf{Im}\, d^1}$$

Proof. Because $B = C \ltimes E$, we have $\mathfrak{r}_B = \mathfrak{r}_C \oplus E$. Also, $B_0 = C_0$ so that

$$\mathsf{Hom}_{B_0^e}(\mathfrak{r}_B, E) = \mathsf{Hom}_{C_0^e}(\mathfrak{r}_C, E) \oplus \mathsf{End}_{C_0^e}(E).$$

Thus, any $f \in \mathsf{Hom}_{B_0^e}(\mathfrak{r}_B, E)$ can be written uniquely as $f = f_1 + f_2$ with $f_1 \in \mathsf{Hom}_{C_0^e}(\mathfrak{r}_C, E)$ and $f_2 \in \mathsf{End}_{C_0^e}(E)$. Let $b = c + e \in \mathfrak{r}_B$, where $c \in \mathfrak{r}_C$ and $e \in E$. Then

$$f(b) = f_1(c) + f_2(e).$$

The statement of the lemma will follow easily from the next three claims.

(1) We first claim that $\mathsf{Im}\, d^1 \subseteq \mathsf{Hom}_{C_0^e}(\mathfrak{r}_C, E)$. Indeed, let $f \in \mathsf{Im}\, d^1$. There exists $e_0 \in E^{B_0}$ such that $f = [e_0, -]$. But then, for every $b = c + e \in \mathfrak{r}_B$ (with $c \in \mathfrak{r}_C$ and $e \in E$) we have $f(b) = [e_0, c] + [e_0, e]$. Now $[e_0, e] = e_0 e - e e_0 = 0$ because $E^2 = 0$. Therefore $f(b) = [e_0, c] = f_1(c)$, that is, $f = f_1$.

(2) Next, we see that $\mathsf{End}_{C^e}(E) \subseteq \mathsf{Ker}\, d^2$. Let indeed $g \in \mathsf{End}_{C^e}(E)$. In particular $g \in \mathsf{End}_{C_0^e}(E)$ so g induces $f \in \mathsf{Hom}_{B_0^e}(\mathfrak{r}_B, E)$ by $f(c + e) = g(e)$ (for $c \in \mathfrak{r}_C$ and $e \in E$). We want to prove that f is a derivation. Let $b, b' \in \mathfrak{r}_B$ be such that $b = c + e$, $b' = c' + e'$ (with $c, c' \in \mathfrak{r}_C$ and $e, e' \in E$). Then

$$f(bb') = f\left((c + e)(c' + e')\right) = g(ec' + ce') = g(e)c' + cg(e') = f(b)c' + cf(b') = f(b)b' + bf(b')$$

because $f(b)e' = ef(b') = 0$.

(3) Finally, we prove that $\mathsf{Ker}\, d^2 \subseteq \mathsf{Hom}_{C_0^e}(\mathfrak{r}_C, E) \oplus \mathsf{End}_{C^e}(E)$. Write $f \in \mathsf{Ker}\, d^2$ as before in the form $f = f_1 + f_2$ with $f_1 \in \mathsf{Hom}_{C_0^e}(\mathfrak{r}_C, E)$ and $f_2 \in \mathsf{End}_{C_0^e}(E)$. We claim that in fact f_2 is a morphism of C-C-bimodules. Let $c \in C$ and $e \in E$, then $ce \in E$ so that $f_2(ce) = f(ce)$. Now, f is a derivation, hence

$$f_2(ce) = f(ce) = f(c)e + cf(e) = f_1(c)e + cf_2(e) = cf_2(e)$$

because $f_1(c)e \in E^2 = 0$. Similarly, $f_2(ec) = f_2(e)c$. This proves that $\mathsf{Ker}\, d^2 \subseteq \mathsf{Hom}_{C_0^e}(\mathfrak{r}_C, E) +$ $\mathsf{End}_{C^e}(E)$. Because $\mathsf{End}_{C^e}(E) \subseteq \mathsf{End}_{C_0^e}(E)$, the sum is direct.

$\square$

We now see that, in general, $\mathsf{HH}^1(B, E)$ depends on the direct decompositions of E as C-C-bimodule. We recall first that, as C-C-bimodule, E is generated by the new arrows. If two new arrows occur in a strongly minimal relation, this means that they are somehow yoked together in E. Direct decompositions of E as C-C-bimodule are studied in [3] from which we import the following result. We include a proof for the benefit of the reader.

4.3. Lemma. *As C-C-bimodule, E decomposes as the direct sum of n nonzero summands.*

Proof. Let $\mathcal{S}_1, \ldots \mathcal{S}_n$ be the equivalence classes of new arrows, with $\mathcal{S}_j = \{\gamma_{j1}, \ldots, \gamma_{js_j}\}$. Further, let E_j be the C-C-bimodule generated by the arrows of $\mathcal{S}_j$, so its elements are of the form $x_j = \sum_{k=1}^{s_j} b_{jk} u_{jk} \gamma_{jk} v_{jk}$ where, as before, the b_{jk} are scalars while u_{jk} and v_{jk} are (classes of) paths. We thus have a natural epimorphism $\eta \colon \bigoplus_{j=1}^n E_j \to E$ given by $\eta(x_1, \ldots, x_n) = \sum_{j=1}^n x_j$. We show that it is also a monomorphism. Let $(x_1, \ldots x_n)$ be a non-zero element of the kernel of η with the additional property that the number of non-zero elements among the x_j is minimal. Thus, we have a relation

$$\sum_{j=1}^n \sum_{k=1}^{s_j} b_{jk} u_{jk} \gamma_{jk} v_{jk} = 0.$$

If $n = 1$ there is nothing to show, so assume there are at least two classes occurring in the relation (that is two values of j). By definition, this relation is not strongly minimal, so there must be a strongly minimal relation involving the same paths

$$\sum_{j=1}^n \sum_{k=1}^{s_j} b'_{jk} u_{jk} \gamma_{jk} v_{jk} = 0.$$

By definition of the equivalence relation $\approx$ only one class appears in the second relation. By subtracting a multiple of this second relation from the first one, we can reduce the number of non-zero terms in the original one, still getting an element of $\mathsf{Ker}\, \eta$, a contradiction. $\square$

Assume $E = E_1 \oplus \cdots \oplus E_n$, with the E_i nonzero. Then the identity on each E_i induces clearly an endomorphism of E as C-C-bimodule. Since these endomorphisms are linearly independent, we get the following corollary.

4.4. Corollary. *We have $\dim_{\Bbbk} \mathsf{HH}^1(B, E) \geqslant n$, and if equality holds, then*

$$\mathsf{Hom}_{C^e}(E_i, E_j) = \begin{cases} \Bbbk & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Write $E = \bigoplus_{j=1}^n E_j$ so that, as vector spaces, we have $\mathsf{End}_{C^e}(E) = \bigoplus_{i,j=1}^n \mathsf{Hom}_{C^e}(E_i, E_j)$. The identity maps on each E_j provide n linearly independent elements in $\mathsf{End}_{C^e}(E)$ which, by

Lemma 4.2 , is contained in $\mathsf{HH}^1(B, E)$. This proves the first statement. If in addition equality holds, we must have $n = \dim_{\Bbbk} \mathsf{End}_{C^e}(E)$, and the conclusion follows.   □

We know that equality occurs under the hypothesis of [5], so we get the following corollary.

4.5. Corollary. *Assume $B \ltimes E$ is such that B is tame or C is constricted, then the indecomposable summands of E as C-C-bimodule are pairwise orthogonal bricks.*

                                                                                                □

Actually, we state:

4.6. Conjecture. *Let C be a tilted algebra, and $E = \mathsf{Ext}_C^2(DC, C)$. Then the indecomposable summands of E as C-C-bimodule are pairwise orthogonal bricks.*

5. An example

Let $\Bbbk$ be a field of characteristic different from 2. Consider the tilted algebra C given by the bound quiver of figure 1a. The relations for C form a strongly minimal set of relations. The
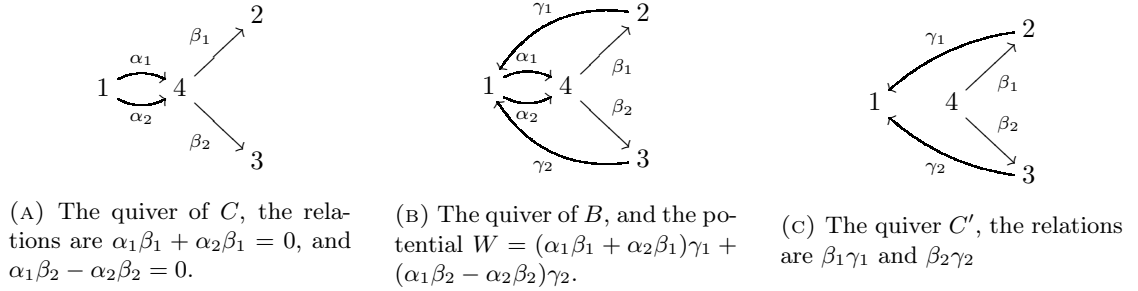

(A) The quiver of C, the relations are $\alpha_1\beta_1 + \alpha_2\beta_1 = 0$, and $\alpha_1\beta_2 - \alpha_2\beta_2 = 0$.

(B) The quiver of B, and the potential $W = (\alpha_1\beta_1 + \alpha_2\beta_1)\gamma_1 + (\alpha_1\beta_2 - \alpha_2\beta_2)\gamma_2$.

(C) The quiver C', the relations are $\beta_1\gamma_1$ and $\beta_2\gamma_2$

FIGURE 1. The cluster-tilted algebra B as relation extension of two tilted algebras C and C'.

corresponding cluster-tilted algebra is B, given by the quiver of figure 1b with potential $W = (\alpha_1\beta_1 + \alpha_2\beta_1)\gamma_1 + (\alpha_1\beta_2 - \alpha_2\beta_2)\gamma_2$. In this case, the partial derivatives with respect to the old arrows are

$$\partial_{\alpha_1} W = \beta_1\gamma_1 + \beta_2\gamma_2, \qquad \partial_{\alpha_2} W = \beta_1\gamma_1 - \beta_2\gamma_2,$$
$$\partial_{\beta_1} W = \gamma_1\alpha_1 + \gamma_1\alpha_2, \qquad \partial_{\beta_2} W = \gamma_2\alpha_1 - \gamma_2\alpha_2.$$

However, these relations do not form a system of strongly minimal relations, since $\partial_{\alpha_1} W + \partial_{\alpha_2} W = 2\beta_1\gamma_1$, and this leads to a monomial relation $\beta_1\gamma_1$. In a similar way we obtain a monomial relation $\beta_2\gamma_2$. This shows that $\gamma_1 \not\approx \gamma_2$, hence there are two equivalence classes. Here the extension bimodule E decomposes as $E_1 \oplus E_2$, with $E_1 = C\gamma_1 C$ and $E_2 = C\gamma_2 C$. Assume $f \in \mathsf{Hom}_{C^e}(E_1, E_2)$. Since $f(\gamma_1) = f(e_2\gamma_1 e_1) = e_2 f(\gamma_1) e_1$ and this lies in E_2, we have $f = 0$. Using similar arguments one gets that E_1 and E_2 are orthogonal bricks.

On the other hand, let C' be the algebra given by the bound quiver of figure 1c. Then B is the relation extension $C' \ltimes E'$. In this case the two new arrows are α_1 and α_2, so that $E' = E_1' \oplus E_2'$ with $E_1' = C'\alpha_1 C'$ and $E_2' = C'\alpha_2 C'$. As vector spaces, we have

$$E_1' = \langle \alpha_1, \alpha_1\beta_2, \gamma_2\alpha_1, \gamma_2\alpha_1\beta_2 \rangle \quad \text{and} \quad E_2' = \langle \alpha_2, \alpha_1\beta_1, \gamma_1\alpha_2, \gamma_1\alpha_2\beta_1 \rangle.$$

Assume $f \in \mathsf{Hom}_{C^e}(E_1', E_2')$. Since $f(\alpha_1) = f(e_1 \gamma_1 e_4) = e_1 f(\alpha_1) e_4$ lies in E_2', there exists a scalar λ such that $f(\alpha_1) = \lambda \alpha_2$. Now, we have

$$0 = f(\alpha_1 \beta_1) = f(\alpha_1)\beta_1 = \lambda \alpha_2 \beta_2$$

and this forces $\lambda = 0$ so that $\mathsf{Hom}_{C^e}(E_1', E_2') = 0$. Using similar arguments one can get that E_1' and E_2' are orthogonal bricks.

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